ONLINE APPENDIX: SOLVING LINEAR DSGE MODELS WITH STRUCTURE PRESERVING DOUBLING METHODS

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1. Pedagogical Scalar Intuition

To illustrate doubling methods and build intuition, consider first the calculation of the geometric series

$$p = \sum_{j=0}^{\infty} \beta^j, \ |\beta| < 1 \tag{1}$$

The solution p can be expressed as the limit of the partial sum

$$p = \frac{1}{1-\beta} = \lim_{k\to\infty} p_k, \ p_k = \sum_{j=0}^{k} \beta^j \tag{2}$$

and p can be recovered by iterating on

$$p_k = 1 + \beta p_{k-1}, \qquad k \geq 0, \qquad p_{-1} = 0. \tag{3}$$

Alternatively, we can also use a doubling algorithm. Consider the $2k-1$'th partial sum

$$p_{2k-1} = \sum_{j=0}^{2k-1} \beta^j = \underbrace{1 + \beta + \beta^2 + \ldots + \beta^{k-1}}_{k \text{ terms}} + \underbrace{\beta^k + \ldots + \beta^{2k-1}}_{k \text{ terms}} = \sum_{j=0}^{k-1} \beta^j + \beta^k \sum_{j=0}^{k-1} \beta^j \tag{4}$$

Iterating on

$$x_{k+1} = x_k + e_k x_k, \qquad e_{k+1} = e_k^2, \qquad k \geq 0, \qquad x_0 = 1, \qquad e_0 = \beta, \tag{5}$$

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gives for the first several terms

$$x_0 = \underbrace{1}_{p_0}, \qquad x_1 = x_0 + e_0 x_0 = \underbrace{1+\beta}_{p_1}, \qquad e_1 = \beta^2 = \beta^{2^1}, \tag{6}$$

$$x_2 = x_1 + e_1 x_1 = 1+\beta+\beta^2\left(1+\beta\right) = \underbrace{1+\beta+\beta^2+\beta^3}_{p_3}, \qquad e_2 = \beta^4 = \beta^{2^2}, \tag{7}$$

$$x_3 = x_2 + e_2 x_2 = 1+\beta+\beta^2+\beta^3+\beta^4\left(1+\beta+\beta^2+\beta^3\right) = \underbrace{\sum_{j=0}^{7}\beta^j}_{p_7}, \qquad e_3 = \beta^8 = \beta^{2^3}. \tag{8}$$

The relation $x_k = p_{2^k-1}$ and the factor 2^k gives the method its name. Clearly x_k will converge more quickly in k to p than p_k.

To highlight some properties of Structure-Preserving Doubling Algorithms (SDAs) that will be important when we extend the above doubling approach to more complicated problems, e.g., for the unilateral quadratic matrix equations (UQME) in (5), we redefine our example of the geometric series in terms of a vector space. Note that since the solution to the geometric series is a fixed point to $p = 1+\beta p$, it is also the fixed point to the subspace problem

$$\mathscr{A}_0\mathscr{X} = \mathscr{X}\beta, \qquad \mathscr{A}_0 = \begin{pmatrix}\beta & 0\\ -1 & 1\end{pmatrix}, \qquad \mathscr{X} = \begin{pmatrix}1\\ p\end{pmatrix}. \tag{9}$$

The triviality of this problem makes the concept of a SDA that we will apply in a more general way more approachable. To tackle (9) by the means of a SDA, we generate a sequence $\{\mathscr{A}_k\}_{k=1}^N$ with $\mathscr{A}_k = \mathscr{A}_{k-1}^2$, such that

$$\mathscr{A}_0\mathscr{X} = \mathscr{X}\beta, \tag{10}$$

$$\mathscr{A}_1\mathscr{X} = \mathscr{A}_0\mathscr{A}_0\mathscr{X} = \mathscr{A}_0\mathscr{X}\beta = \mathscr{X}\beta^2, \tag{11}$$

$$\vdots$$

$$\mathscr{A}_{k+1}\mathscr{X} = \mathscr{A}_k\mathscr{A}_k\mathscr{X} = \mathscr{A}_k\mathscr{X}\beta^{2^k} = \mathscr{X}\beta^{2^{k+1}}. \tag{12}$$

At this simple example we can see the two essential properties of the SDA: "doubling" while "preserving structure". On the one hand, in each iteration, the eigenvalues of the matrix pencil $P_{I\mathscr{A}_k}(z) = Iz - \mathscr{A}_k$ are squared. Furthermore, since the exponent of β is doubled in each iteration, the righthand side of (12) will eventually converges to a vector of zeros. On the other hand, the eigenspace and the structure of $\mathscr{A}_0$, which is lower triangular, are preserved in each iteration. This enables us to reduce the complexity of the problem, defining the recursion in the entries of $\mathscr{A}_k$ instead of $\mathscr{A}_k$ in its entirety

$$\mathscr{A}_{k+1} = \begin{pmatrix} e_{k+1} & 0\\ -x_{k+1} & 1\end{pmatrix} = \begin{pmatrix} e_k & 0\\ -x_k & 1\end{pmatrix}\begin{pmatrix} e_k & 0\\ -x_k & 1\end{pmatrix} = \begin{pmatrix} e_k^2 & 0\\ -(x_k + e_k x_k) & 1\end{pmatrix}. \tag{13}$$

Note that resulting recursion is the same as in (5). Besides the triangular structure, we can also use the fact that the second diagonal element of $\mathscr{A}_k$ is 1 for all k to show that x_k is an approximation to p with approximation error $|p - x_k| = \beta^{2^k}|p|$.

Foreshadowing the general application, note that the SDA presented in this simple example is a special case of the SDA based on the First Standard Form (SF1) we present in the main text. While we will have to adjust the above version of a SDA to account for the more complex structure of the UQME in (5), the basic concepts applied above will remain same: We rearrange the problem in terms of a subspace problem that satisfies either SF1 or Second Standard Form (SF2). The specific structure of the problem will then allow us to generate a sequence of matrix pencils, squaring the corresponding eigenvalues in each step, while preserving the structure of the problem - hence the name SDAs.

For (13) in the main text, we see that the structure of the matrix pencil $\mathscr{A}_0 - \lambda\mathscr{B}_0$ is quite similar to the one in our example of the geometric series, except that the matrices $\mathscr{A}_0$ and $\mathscr{B}_0$ are now block triangular.[1]

[1]Note that in our example of the geometric series $\mathscr{B}_0$ reduced to the identity matrix.

2. QZ Implementation

The QZ decomposition provides unitary Q and Z and upper triangular S and T with $Q^*\mathscr{B}Z = S$ and $Q^*\mathscr{A}Z = T$, where * indicates the complex conjugation of Q that delivers its inverse by virtue of it being a unitary matrix and where the eigenvalues of the matrix pencil $P_{\mathscr{B}\mathscr{A}}(z) = \mathscr{B}z - \mathscr{A}$, $\rho(P_{\mathscr{B}\mathscr{A}}) = \rho(P_{ST}) = \{t_{ii}/s_{ii}$, if $s_{ii} \neq 0$; ∞, if $s_{ii} = 0$; $\emptyset$, if $s_{ii} = t_{ii} = 0$; $i = 1,\ldots,2n_y\}$,[2] can be ordered arbitrarily to form

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} P \quad \text{with} \quad Z^*\mathscr{X} = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}. \tag{14}$$

We assume the decomposition is ordered so that the unstable eigenvalues are in the lower right blocks of S and T (hence S_{22} and T_{22}), wherefor the lower block equation gives

$$T_{22}W_2 = S_{22}W_2P \Rightarrow W_2 = T_{22}^{-1}S_{22}W_2P \tag{15}$$

as the eigenvalues of $T_{22}^{-1}S_{22}$ are inside the unit circle and P is a (semi) stable solvent we can iterate

$$W_2 = T_{22}^{-1}S_{22}W_2P = \left(T_{22}^{-1}S_{22}\right)^2 W_2P^2 = \left(T_{22}^{-1}S_{22}\right)^j W_2P^j \underset{j\to\infty}{\rightarrow} 0 \tag{16}$$

for a unit root stable P. Using the definition of W_2

$$0 = W_2 = Z_{21}^* + Z_{22}^*P \Rightarrow P = -Z_{22}^{*\,-1}Z_{21}^* = Z_{21}Z_{11}^{-1} \tag{17}$$

The equivalence $Z_{22}^{*\,-1}Z_{21}^* = -Z_{21}Z_{11}^{-1}$ follow from the properties of unitary matrices and $Z_{21}Z_{11}^{-1} = Q_{11}S_{11}^{-1}T_{11}Q_{11}^{-1}$ from the first block rows of $\mathscr{A}$ and $\mathscr{B}$ in (6) and upper triangularity of S and T. From $Q_{11}S_{11}^{-1}T_{11}Q_{11}^{-1}$, it follows that the recursion in P is indeed stable from the ordering of the eigenvalues above, i.e. the eigenvalues of the upper left block of the generalized Schur decomposition, $det(S_{11}\lambda - T_{11}) = 0$, are inside the unit circle. So the QZ decomposition applied to our matrix pencil will recover the unique (semi) stable solvent P if it exists consistent with our assumption 1. Importantly, we did not solve the problem by working directly with the pencil $P_{\mathscr{B}\mathscr{A}}(z) = \mathscr{B}z - \mathscr{A}$ but first transformed the problem unitarily with Q and Z.

Note that we can recycle the QZ factorization used above and solve simultaneously for its dual equation – the original UQME in (5) "in reverse"

$$0 = CP_d^2 + BP_d + A \tag{18}$$

To see this, note that we can derive the first companion linearization of the dual problem (18) as

$$\mathscr{B}\mathscr{Y} = \mathscr{A}\mathscr{Y}\mathscr{N}, \quad \text{with} \quad \mathscr{Y} = \begin{pmatrix} P_d \\ I \end{pmatrix}, \quad \mathscr{N} = P_d. \tag{19}$$

Hence, we can just reuse QZ decomposition of the primal problem to receive

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} P_d = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \quad \text{with} \quad Z^*\mathscr{Y} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{20}$$

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[2] If both $s_{ii} = t_{ii} = 0$ for some i, then any $z \in \mathbb{C}$ will satisfy $|\mathscr{B}z - \mathscr{A}| = 0$. This is the mundane singularity of King and Watson (1998) that reflects a poorly specified model, e.g., a repeated equation, and the definition in the text can be interpreted to ruling out such misspecifications by limiting the analysis to regular pencils.

To obtain P_d we use a guess and verify strategy. Assuming that $V_1 = 0$, we get from the definition of $\begin{bmatrix} V_1' & V_2' \end{bmatrix}'$ that

$$P_d = Z_{11}V_1 + Z_{12}V_2 = Z_{12}V_2, \quad \text{and} \quad I = Z_{21}V_1 + Z_{22}V_2 = Z_{22}V_2. \tag{21}$$

Consequently, $V_2 = Z_{22}^{-1}$ and P_d must take the form $P_d = Z_{12}Z_{22}^{-1}$. Furthermore, we can verify from the lower block equation of (20)

$$T_{22}V_2P_d = S_{22}V_2 \quad \Rightarrow \quad P_d = Z_{22}T_{22}^{-1}S_{22}Z_{22}^{-1} \tag{22}$$

that the resulting P_d is indeed stable. Now we need to verify that the constructed solution P_d actually implies $V_1 = 0$. We do so using the upper block equation of (20), which yields

$$T_{11}V_1P_d + T_{12}V_2P_d = S_{11}V_1 + S_{12}V_2 \quad \Longleftrightarrow \quad T_{11}V_1P_d + T_{12}Z_{22}^{-1}P_d = S_{11}V_1 + S_{12}Z_{22}^{-1}, \tag{23}$$

Note that from $\mathscr{B}Z = QS$ and $\mathscr{A}Z = QT$ and the first row blocks of $\mathscr{A}$ and $\mathscr{B}$ follows that

$$Z_{12} = Q_{11}S_{12} + Q_{12}S_{22} \quad \Rightarrow \quad S_{12} = Q_{11}^{-1}Z_{12} - Q_{11}^{-1}Q_{12}S_{22}, \tag{24}$$

$$Z_{22} = Q_{11}T_{12} + Q_{12}T_{22} \quad \Rightarrow \quad T_{12} = Q_{11}^{-1}Z_{22} - Q_{11}^{-1}Q_{12}T_{22}. \tag{25}$$

If we now substitute (22), (24) and (25) into (23) we get

$$T_{11}V_1P_d = S_{11}V_1 \quad \Rightarrow \quad V_1 = S_{11}^{-1}T_{11}V_1P_d. \tag{26}$$

As the eigenvalues of $S_{11}^{-1}T_{11}$ and P_d are inside unit circle we can iterate

$$V_1 = S_{11}^{-1}T_{11}V_1P_d = \left(S_{11}^{-1}T_{11}\right)^2 V_1P_d^2 = \left(S_{11}^{-1}T_{11}\right)^j V_1P_d^j \underset{j\to\infty}{\to} 0. \tag{27}$$

This also shows that a stable dual solution P_d exists under the Blanchard and Kahn (1980) conditions, since these imply that the matrices Z_{11}, Z_{22}, and Q_{11} exist and are non-singular.

3. Derivations and Pseudocode for Reductions from Section 4.1

In this section we provide detailed derivations as well as pseudocode for the cyclic and the logarithmic reduction.

3.1. Cyclic Reduction

As described in the main text cyclic reduction computes the sequences $\{A_k\}_{k=0}^{\infty}$, $\{B_k\}_{k=0}^{\infty}$, $\{C_k\}_{k=0}^{\infty}$, and $\{\widehat{B}_k\}_{k=0}^{\infty}$ with

$$A_{k+1} = -A_k\, B_k^{-1}\, A_k, \qquad A_0 = A \tag{28}$$

$$B_{k+1} = B_k - A_k\, B_k^{-1}\, C_k - C_k\, B_k^{-1}\, A_k, \qquad B_0 = B \tag{29}$$

$$C_{k+1} = -C_k\, B_k^{-1}\, C_k, \qquad C_0 = C \tag{30}$$

$$\widehat{B}_{k+1} = \widehat{B}_k - A_k\, B_k^{-1}\, C_k \qquad \widehat{B}_0 = B \tag{31}$$

to create a sequence of UQMEs with

$$0 = A_k \mathscr{M}_k^2 + B_k \mathscr{M}_k + C_k, \qquad 0 = A_k \mathscr{M}_k \mathscr{M} + \widehat{B}_k \mathscr{M} + C, \tag{32}$$

where $\mathscr{M}_k = \mathscr{M}^{2^k}$. To see this, note that both equations in (32) hold for $k = 0$. Further, assuming that they hold for an arbitrary $k \geq 0$, multiply the left equation of (32) from the right by $\mathscr{M}$, $\mathscr{M}_k$ and $\mathscr{M}_k^2$ to obtain

$$0 = A_k \mathscr{M}_k^2 \mathscr{M} + B_k \mathscr{M}_k \mathscr{M} + C_k \mathscr{M}, \tag{33}$$

$$0 = A_k \mathscr{M}_k^3 + B_k \mathscr{M}_k^2 + C_k \mathscr{M}_k, \tag{34}$$

$$0 = A_k \mathscr{M}_k^4 + B_k \mathscr{M}_k^3 + C_k \mathscr{M}_k^2. \tag{35}$$

If we now add $C_k \mathscr{M}_k$, $A_k \mathscr{M}_k \mathscr{M}$, and $A_k \mathscr{M}_k^3$ to (32), (33), and (35) multiplied by $-C_k B_k^{-1}$, $-A_k B_k^{-1}$, and $-A_k B_k^{-1}$, respectively, we get

$$C_k \mathscr{M}_k = -C_k B_k^{-1} A_k\, \mathscr{M}_k^2 - C_k B_k^{-1} C_k, \tag{36}$$

$$A_k \mathscr{M}_k \mathscr{M} = -A_k B_k^{-1} A_k\, \mathscr{M}_k^2 \mathscr{M} - A_k B_k^{-1} C_k\, \mathscr{M}, \tag{37}$$

$$A_k \mathscr{M}_k^3 = -A_k B_k^{-1} A_k\, \mathscr{M}_k^4 - A_k B_k^{-1} C_k\, \mathscr{M}_k^2. \tag{38}$$

Finally, substituting (36) and (38) in (34) as well as (37) in the right equation of (32)

$$0 = A_{k+1}\, \mathscr{M}_{k+1}^2 + B_{k+1}\, \mathscr{M}_{k+1} + C_{k+1}, \qquad 0 = A_{k+1}\, \mathscr{M}_{k+1} \mathscr{M} + \widehat{B}_{k+1} \mathscr{M} + C. \tag{39}$$

Hence, both equations of (32) hold in $k+1$ as well. Furthermore, assuming that $\lim_{k\to\infty} \widehat{B}_k^{-1}$ exists and that $\lim_{k\to\infty} \widehat{B}_k^{-1} A_k \mathscr{M}_k \mathscr{M} = 0$ we can express the solvent P as $P = -\lim_{k\to\infty} \widehat{B}_k^{-1} C$. Algorithm 1 summarizes the cyclic reduction.

Algorithm 1: Cyclic Reduction

Given: A, B, C, and a convergence criterion ϵ

Set A_0, B_0, C_0, and $\widehat{B}_0$ according to

$$A_0 = A, \qquad B_0 = B, \qquad C_0 = C, \qquad \widehat{B}_0 = B$$

While *criterion*($\widehat{B}_k$) $> \epsilon$ **do**

Set $A_{k+1} = -A_k B_k^{-1} A_k$ and $B_{k+1} = B_k - A_k B_k^{-1} C_k - C_k B_k^{-1} A_k$

Set $C_{k+1} = -C_k B_k^{-1} C_k$ and $\widehat{B}_{k+1} = \widehat{B}_k - A_k B_k^{-1} C_k$

Advance $k = k+1$

end

Return: $-\widehat{B}_k^{-1} C_0$

3.2. Logarithmic Reduction

As discussed in the main text the logarithmic reduction by Latouche and Ramaswami (1993) uses the same divide-and-conquer strategy as the cyclic reduction to obtain $\{H_k\}_{k=0}^{\infty}$, $\{L_k\}_{k=0}^{\infty}$, $\{\widehat{H}_k\}_{k=0}^{\infty}$, and $\{\widehat{L}_k\}_{k=0}^{\infty}$ with

$$H_{k+1} = (I - H_k L_k - L_k H_k)^{-1} H_k^2, \qquad H_0 = -B^{-1} A \tag{40}$$

$$L_{k+1} = (I - H_k L_k - L_k H_k)^{-1} L_k^2, \qquad L_0 = -B^{-1} C, \tag{41}$$

$$\widehat{H}_{k+1} = \widehat{H}_k H_{k+1}, \qquad \widehat{H}_0 = -B^{-1} A, \tag{42}$$

$$\widehat{L}_{k+1} = \widehat{L}_k + \widehat{H}_k L_{k+1}, \qquad \widehat{L}_0 = -B^{-1} C, \tag{43}$$

to create a sequence of UQMEs similarly to the cyclic reduction above with

$$0 = H_k \mathcal{M}_k^2 - \mathcal{M}_k + L_k. \qquad 0 = \widehat{H}_k \mathcal{M}_{k+1} - \mathcal{M} + \widehat{L}_k \tag{44}$$

As pointed out by Bini et al. (2005, Theorem 7.5) and discussed in the main text, it follows via induction that for all $k \geq 0$

$$H_k = -B_k^{-1} A_k, \qquad L_k = -B_k^{-1} C_k,$$

so that the left equations (44) follows directly from multiplying (32) from the left by $-B_k^{-1}$. Consequently, there is also a link between the SDA for SF2 with $P_0 = 0$ and the logarithmic reduction. In detail, we get for all $k \geq 0$ that

$$H_k = \left(X_k^{\dagger} - Y_k^{\dagger}\right)^{-1} F_k^{\dagger} \qquad L_k = \left(X_k^{\dagger} - Y_k^{\dagger}\right)^{-1} E_k^{\dagger}$$

The right equation of (44) also follows by induction: in $k=0$ it follows directly from multiplying (5) from the left by B^{-1}. Assuming that it holds for an arbitrary $k \geq 0$, we get

$$\begin{aligned} 0 = \widehat{H}_k \mathscr{M}_{k+1} - \mathscr{M} + \widehat{L}_k &= \widehat{H}_k H_{k+1} \mathscr{M}_{k+1}^2 + \widehat{H}_k L_{k+1} - \mathscr{M} + \widehat{L}_k, \\ &= \widehat{H}_{k+1} \mathscr{M}_{k+1}^2 - \mathscr{M} + \widehat{L}_{k+1}, \end{aligned} \tag{45}$$

where we can use the fact that from the left equation in (44) it follows that $\mathscr{M}_{k+1} = H_{k+1}\mathscr{M}_{k+1}^2 + L_{k+1}$. Assuming $\lim\limits_{k\to\infty} \widehat{H}_k \mathscr{M}_{k+1} = 0$ we can express the stable solvent of as

$$P = \lim_{k\to\infty} \widehat{L}_k. \tag{46}$$

Algorithm 2 summarizes the logarithmic reduction.

Algorithm 2: Logarithmic Reduction

Given: A, B, C, and a convergence criterion ϵ

Set L_0, H_0, $\widehat{L}_0$, $\widehat{H}_0$ according to

$$L_0 = -B^{-1}C, \qquad H_0 = -B^{-1}A, \qquad \widehat{L}_0 = -B^{-1}C, \qquad \widehat{H}_0 = -B^{-1}A$$

While *criterion*($\widehat{L}_k$) $> \epsilon$ **do**

- Set $U_k = (I - H_k L_k - L_k H_k)$
- Set $L_{k+1} = U_k^{-1} L_k^2$ and $H_{k+1} = U_k^{-1} H_k^2$
- Set $\widehat{L}_{k+1} = \widehat{L}_k + \widehat{H}_k L_{k+1}$ and $\widehat{H}_{k+1} = \widehat{H}_k H_{k+1}$
- Advance $k = k+1$

end

Return: $\widehat{L}_k$

4. Initial guess P_0 for $AP^2 + BP + C = 0$

Suppose we want to minimize the squared Frobenius norm $\|R(P_0)\|_F^2$ of the residuals

$$R(P_0) = AP_0^2 + BP_0 + C,$$

where we restrict P_0 to

$$P_0 = \operatorname{diag}(p_1, \ldots, p_n), \quad \text{with} \quad p1, \ldots, p_n \in [-\rho, \rho], \; \rho \in [0,1).$$

Consequently, we have

$$\begin{aligned} \|R(P_0)\|_F^2 &= \sum_{i=1}^n \sum_{j=1}^n \left(a_{ij} p_j^2 + b_{ij} p_j + c_{ij}\right)^2 \\ &= \sum_{j=1}^n \sum_{i=1}^n a_{ij}^2 \, p_j^4 + 2a_{ij} b_{ij} \, p_j^3 + \left(b_{ij}^2 + 2a_{ij} c_{ij}\right) \, p_j^2 + 2b_{ij} c_{ij} \, p_j + c_{ij}^2. \end{aligned}$$

Further, denoting the j-th column of R, A, B, and C as r_j, a_j, b_j, and c_j, respectively, we may write

$$r_j(p_j) = \tilde{a}\ p_j^4 + \tilde{b}\ p_j^3 + \tilde{c}\ p_j^2 + \tilde{d}\ p_j + \tilde{e} \tag{47}$$

with

$$\begin{aligned}
\tilde{a} &= a_j^T\ a_j \\
\tilde{b} &= 2\ a_j^T\ b_j, \\
\tilde{c} &= b_j^T\ b_j + 2\ a_j^T\ c_j, \\
\tilde{d} &= 2\ b_j^T\ c_j, \\
\tilde{e} &= c_j^T\ c_j.
\end{aligned}$$

Differentiating r_j with respect to p_j yields

$$r_j'(p_j) = 4\ \tilde{a}\ p_j^3 + 3\ \tilde{b}\ p_j^2 + 2\ \tilde{c}\ p_j + \tilde{d}$$

Hence, an interior solution for p_j must satisfy $r_j'(p_j) = 0$, so that we can obtain the $p_j \in [-\rho, \rho]$ that minimizes $r_j(p_j)$ as

$$p_j^* = \underset{p_j \in g_j \cap \{-\rho, \rho\}}{\operatorname{argmin}}\ r_j(p_j)$$

where g_j is the set of all real roots of $r_j'(p_j)$ in $[-\rho, \rho]$, i.e.,

$$g_j = \left\{ p_j \in [-\rho, \rho] \subset \mathbb{R}\ :\ r_j'(p_j) = 0 \right\}, \quad |g_j| \le 3.$$

Algorithm 3 summarizes the proceeding to obtain the diagonal elements of P_0.

Algorithm 3: Initial Guess – j-th diagonal element of P_0

Given: a_j, b_j, c_j, and ρ

Set $\tilde{a} = a_j^T\ a_j$
Set $\tilde{b} = 2\ a_j^T\ b_j$
Set $\tilde{c} = b_j^T\ b_j + 2\ a_j^T\ c_j$
Set $\tilde{d} = 2\ b_j^T\ c_j$
Set $\tilde{e} = c_j^T\ c_j$
Define $r(p_j) := \tilde{a}\ p_j^4 + \tilde{b}\ p_j^3 + \tilde{c}\ p_j^2 + \tilde{d}\ p_j + \tilde{e}$
Define $r'(p_j) := 4\ \tilde{a}\ p_j^3 + 3\ \tilde{b}\ p_j^2 + 2\ \tilde{c}\ p_j + \tilde{d}$
Obtain all $p_j \in g = \left\{ x \in [-\rho, \rho] \subset \mathbb{R}\ :\ r'(x) = 0 \right\}$

Return: $\underset{p_j \in g_j \cap \{-\rho, \rho\}}{\operatorname{argmin}}\ r(p_j)$

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5. Detailed Dynare Topology

Here we summarize the details in the matrix quadratic that follows from the typology of variables from Dynare as laid out in Villemot (2011). See Meyer-Gohde and Saecker (2024) for details.

Subdividing the system of equations in accordance with the QR decomposition yields

$$\underbrace{\begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++}\end{array} \\ \begin{array}{c} n^s\\ n^{--}\\ n^m\\ n^{++}\end{array} & \left[\begin{array}{cc|c} \mathbf{0} & \mathbf{0} & \underset{n^s\times n^+}{\breve{\mathbf{A}}^{+}} \\ \hline \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \underset{n^d\times n^+}{\tilde{\mathbf{A}}^{+}} \\ \mathbf{0} & \mathbf{0} & \end{array}\right]\end{array}}_{\underset{n\times n}{A}} \underset{n\times n}{\mathbf{P}^2} + \underbrace{\begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++}\end{array} \\ \begin{array}{c} n^s\\ n^{--}\\ n^m\\ n^{++}\end{array} & \left[\begin{array}{c|c} \breve{\mathbf{A}}^{0s} & \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}} \\ \hline \mathbf{0} & \\ \mathbf{0} & \underset{n^d\times n^d}{\tilde{\mathbf{A}}^{0}} \\ \mathbf{0} & \end{array}\right]\end{array}}_{\underset{n\times n}{B}} \underset{n\times n}{\mathbf{P}} + \underbrace{\begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++}\end{array} \\ \begin{array}{c} n^s\\ n^{--}\\ n^m\\ n^{++}\end{array} & \left[\begin{array}{c|c|c} \mathbf{0} & \underset{n^s\times n^-}{\breve{\mathbf{A}}^{-}} & \mathbf{0} \\ \hline \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & \underset{n^d\times n^-}{\tilde{\mathbf{A}}^{-}} & \mathbf{0} \\ \mathbf{0} & & \mathbf{0} \end{array}\right]\end{array}}_{\underset{n\times n}{C}} = \underset{n\times n}{\mathbf{0}}$$

where n^d is the number of dynamic variables, the sum of number of purely backward-looking, n^{--}, mixed n^m, and purely forward-looking variables, n^{++}. The number of forward-looking variables, n^+, is the sum of the number of mixed, n^m, and purely forward-looking variables, n^{++}, and the number of backward-looking variables, n^-, is the sum of the number of purely backward-looking, n^{--} and mixed variables n^m. Hence, the number of endogenous variables is the sum of the number of static, n^s, and dynamic variables, n^d, or the sum of the number of static, n^s, purely backward-looking, n^{--}, mixed n^m, and purely forward-looking variables, n^{++}. The dimensions satisfy the following

$$n^d = n^{--} + n^m + n^{++}, \quad n^+ = n^m + n^{++}, \quad n^- = n^{--} + n^m, \quad n = n^s + n^d = n^s + n^{--} + n^m + n^{++}$$

The transition matrix, P, from (4) that solves the matrix equation (5) can be subdivided in accordance to Dynare's typology as

$$\mathbf{P} = \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++}\end{array} \\ \begin{array}{c} n^s\\ n^{--}\\ n^m\\ n^{++}\end{array} & \left[\begin{array}{c|c|c|c} \mathbf{P}_{s,s} & \mathbf{P}_{s,--} & \mathbf{P}_{s,m} & \mathbf{P}_{s,++} \\ \hline \mathbf{P}_{--,s} & \mathbf{P}_{--,--} & \mathbf{P}_{--,m} & \mathbf{P}_{--,++} \\ \hline \mathbf{P}_{m,s} & \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{P}_{m,++} \\ \hline \mathbf{P}_{++,s} & \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{P}_{++,++} \end{array}\right]\end{array} = \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++}\end{array} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{P}_{\bullet,s} & \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} & \mathbf{P}_{\bullet,++} \end{array}\right]\end{array} = \begin{array}{cc} & n \\ \begin{array}{c} n^s\\ n^{--}\\ n^m\\ n^{++}\end{array} & \left[\begin{array}{c} \mathbf{P}_{s,\bullet} \\ \hline \mathbf{P}_{--,\bullet} \\ \hline \mathbf{P}_{m,\bullet} \\ \hline \mathbf{P}_{++,\bullet} \end{array}\right]\end{array}$$

The matrix quadratic can be expressed as

$$\begin{aligned} \mathbf{M}(\underset{n\times n}{\mathbf{P}}) &= \underset{n\times n}{\mathbf{A}}\,\mathbf{P}^2 + \underset{n\times n}{\mathbf{B}}\,\mathbf{P} + \underset{n\times n}{\mathbf{C}} \\ &= \underbrace{(\mathbf{AP}+\mathbf{B})}_{\equiv\mathbf{G}}\mathbf{P} + \mathbf{C} \end{aligned}$$

For a solvent P of the matrix quadratic, taking the structure of C from the Dynare typology above into account yields

$$\mathbf{M}(\mathbf{P}) = 0 = \mathbf{G}\mathbf{P} + \mathbf{C}$$

$$= \mathbf{G} \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{P}_{\bullet,s} & \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} & \mathbf{P}_{\bullet,++} \end{array}\right] \end{array} + \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} & \left[\begin{array}{c|c|c} \mathbf{0} & \underset{n^s\times n^-}{\breve{\mathbf{A}}^-} & \mathbf{0} \\ \hline \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & \underset{n^d\times n^-}{\tilde{\mathbf{A}}^-} & \mathbf{0} \\ \mathbf{0} & & \mathbf{0} \end{array}\right] \end{array}$$

Following Meyer-Gohde and Saecker (2024), who apply corollary 4.5 of Lan and Meyer-Gohde (2014), if P is the unique solvent of $M(P)$ stable with respect to the closed unit circle, $\mathbf{G}$ has full rank and hence the columns of P associated with nonzero columns in C, the static and forward-looking variables are zero $\rightarrow$ $\mathbf{P}_{\bullet,s} = \underset{n\times n^s}{\mathbf{0}}$, $\mathbf{P}_{\bullet,++} = \underset{n\times n^{++}}{\mathbf{0}}$, whence $\mathbf{P}$ is $\mathbf{P} = \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} & \mathbf{0} \end{array}\right] \end{array}$ and $\mathbf{M}(\mathbf{P}) = \left[\begin{array}{cccc} \underset{n\times n^s}{\mathbf{0}} & \underset{n\times n^{--}}{\mathbf{M}(\mathbf{P})^{--}} & \underset{n\times n^m}{\mathbf{M}(\mathbf{P})^m} & \underset{n\times n^{++}}{\mathbf{0}} \end{array}\right]$. Consequentially, the first n^s rows of the matrix quadratic, taking $\begin{array}{cc} & n \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} & \left[\begin{array}{c} \mathbf{P}_{--,\bullet} \\ \hline \mathbf{P}_{m,\bullet} \\ \hline \mathbf{P}_{++,\bullet} \end{array}\right] \end{array}$ as given, yield $\begin{array}{cc} & \begin{array}{cc} n^{--} & n^m \end{array} \\ n^s & \left[\begin{array}{c|c} \mathbf{P}_{s,--} & \mathbf{P}_{s,m} \end{array}\right] \end{array}$ as

$$\begin{array}{cc} & \begin{array}{cc} n^{--} & n^m \end{array} \\ n^s & \left[\begin{array}{c|c} \mathbf{P}_{s,--} & \mathbf{P}_{s,m} \end{array}\right] \end{array} = -\left[\underset{n^s\times n^s}{\breve{\mathbf{A}}^{0s}}\right]^{-1} \left(\underset{n^s\times n^+}{\breve{\mathbf{A}}^+} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^m \end{array} \\ \begin{array}{c} n^m \\ n^{++} \end{array} & \left[\begin{array}{c|c} \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} \end{array}\right] \end{array} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^m \end{array} \\ \begin{array}{c} n^{--} \\ n^m \end{array} & \left[\begin{array}{c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \end{array}\right] \end{array} \right.$$

$$\left. + \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^m \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} & \left[\begin{array}{c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} \end{array}\right] \end{array} + \underset{n^s\times n^-}{\breve{\mathbf{A}}^-} \right)$$

and the first n^s rows of P are $\underset{n^s\times n}{\mathbf{P}_{s,\bullet}} = \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ n^s & \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{P}_{s,--} & \mathbf{P}_{s,m} & \mathbf{0} \end{array}\right] \end{array}$.

The last n^d columns and rows of P solve the reduced matrix quadratic equation

$$
\begin{array}{c@{}c} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|cc} \mathbf{0} & & \\ \mathbf{0} & \underset{n^d\times n^+}{\tilde{\mathbf{A}}^+} & \\ \mathbf{0} & & \end{array}\right] \end{array}
\underbrace{\begin{array}{c@{}c} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} & \mathbf{P}_{--,++} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{P}_{m,++} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{P}_{++,++} \end{array}\right] \end{array}}_{\underset{n^d\times n^d}{\tilde{\mathbf{P}}}}
\cdot \underset{n^d\times n^d}{\tilde{\mathbf{P}}} + \underset{n^d\times n^d}{\tilde{\mathbf{A}}^0}\ \underset{n^d\times n^d}{\tilde{\mathbf{P}}}
$$

$$
+ \begin{array}{c@{}c} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{cc|c} & & \mathbf{0} \\ \hline & \underset{n^d\times n^-}{\tilde{\mathbf{A}}^-} & \mathbf{0} \\ \hline & & \mathbf{0} \end{array}\right] \end{array}
$$

$$
= \underset{n^d\times n^d}{\tilde{\mathbf{M}}(\tilde{\mathbf{P}})} = \begin{array}{c@{}c} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ n^d & \left[\begin{array}{c|c|c} \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{--} & \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{m} & \mathbf{0} \end{array}\right] \end{array} = \underset{n^d\times n^d}{\mathbf{0}}
$$

Recalling that $\mathbf{P}_{\bullet,++} = \underset{n\times n^{++}}{\mathbf{0}}$, $\tilde{\mathbf{P}}$ can be reduced and two submatrices $\overline{\mathbf{P}}$ and $\widehat{\mathbf{P}}$ defined via

$$
\tilde{\mathbf{P}} = \begin{array}{c@{}c} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} & \mathbf{P}_{--,++} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{P}_{m,++} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{P}_{++,++} \end{array}\right] \end{array}
= \begin{array}{c@{}c} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} & \mathbf{0} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{0} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{0} \end{array}\right] \end{array}
\equiv \begin{array}{c@{}c} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c} \underset{n^{--}\times n^-}{\overline{\mathbf{P}}} & \mathbf{0} \\ \hline \underset{n^{+}\times n^-}{\widehat{\mathbf{P}}} & \mathbf{0} \\ & \mathbf{0} \end{array}\right] \end{array}
$$

where $\underset{n^{--}\times n^-}{\overline{\mathbf{P}}} \equiv \begin{array}{c@{}c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ n^{--} & \left[\begin{array}{c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \end{array}\right] \end{array}$ and $\underset{n^{+}\times n^-}{\widehat{\mathbf{P}}} \equiv \begin{array}{c@{}c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c} \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} \end{array}\right] \end{array}$ allow the matrix quadratic to be written as

$$
\left(
\begin{array}{c@{}c} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|cc} \mathbf{0} & & \\ \mathbf{0} & \underset{n^d\times n^+}{\tilde{\mathbf{A}}^+} & \\ \mathbf{0} & & \end{array}\right] \end{array}
\begin{array}{c@{}c} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c} \underset{n^{--}\times n^-}{\overline{\mathbf{P}}} & \mathbf{0} \\ \hline \underset{n^{+}\times n^-}{\widehat{\mathbf{P}}} & \mathbf{0} \\ & \mathbf{0} \end{array}\right] \end{array}
+ \underset{n^d\times n^d}{\tilde{\mathbf{A}}^0}
\right)
\begin{array}{c@{}c} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c} \underset{n^{--}\times n^-}{\overline{\mathbf{P}}} & \mathbf{0} \\ \hline \underset{n^{+}\times n^-}{\widehat{\mathbf{P}}} & \mathbf{0} \\ & \mathbf{0} \end{array}\right] \end{array}
$$

$$
+ \underset{n^d\times n^-}{\tilde{\mathbf{A}}^-} = \underset{n^d\times n^d}{\tilde{\mathbf{M}}(\tilde{\mathbf{P}})} = \begin{array}{c@{}c} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ n^d & \left[\begin{array}{c|c|c} \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{--} & \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{m} & \mathbf{0} \end{array}\right] \end{array} = \underset{n^d\times n^d}{\mathbf{0}}
$$

6. Detailed Dynare Topology - SDA for SF1

Beginning with Algorithm 1 with an initial guess $P_0 = 0$, the initial values

$$
X_0 = -B^{-1}C, \qquad Y_0 = -B^{-1}A, \qquad E_0 = -B^{-1}C, \qquad F_0 = -B^{-1}A
$$

can be written as

$$X_0=-B^{-1}C=\left(\underset{n^d\times n^d}{\tilde{\mathbf{A}}^0}\right)^{-1}\begin{array}{cc} & \begin{array}{cc} n^- & n^{++}\end{array}\\ n^d & \left[\begin{array}{c|c}\underset{n^d\times n^-}{\tilde{\mathbf{A}}^-} & \mathbf{0}\end{array}\right]\end{array}=\left[\begin{array}{c|c}\underset{n^d\times n^-}{(\tilde{\mathbf{A}}^0)^{-1}\tilde{\mathbf{A}}^-} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]=\left[\begin{array}{c|c}\underset{n^d\times n^-}{\overline{X_0}} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]=\left[\begin{array}{c|c}\underset{n^d\times n^-}{\overline{E_0}} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]=E_0$$

and

$$Y_0=-B^{-1}A=\left(\underset{n^d\times n^d}{\tilde{\mathbf{A}}^0}\right)^{-1}\begin{array}{cc} & \begin{array}{cc} n^{--} & n^{+}\end{array}\\ n^d & \left[\begin{array}{c|c}\mathbf{0} & \underset{n^d\times n^+}{\tilde{\mathbf{A}}^+}\end{array}\right]\end{array}=\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{(\tilde{\mathbf{A}}^0)^{-1}\tilde{\mathbf{A}}^+}\end{array}\right]=\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{\overline{Y_0}}\end{array}\right]=\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{\overline{F_0}}\end{array}\right]=F_0$$

Now proceeding by induction and assuming that X_k, Y_k, E_k, and F_k have the same dimensions (i.e., zero and non zero), we will show this holds for X_{k+1}, Y_{k+1}, E_{k+1}, and F_{k+1}. Beginning with E_{k+1},

$$E_{k+1}=E_k(I-Y_kX_k)^{-1}E_k=E_k(I-Y_kX_k)^{-1}\left[\begin{array}{c|c}\underset{n^d\times n^-}{\overline{E_k}} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]=\left[\begin{array}{c|c}\underset{n^d\times n^-}{E_k(I-Y_kX_k)^{-1}\overline{E_k}} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]=\left[\begin{array}{c|c}\underset{n^d\times n^-}{\overline{E_{k+1}}} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]$$

which has the same zero and non zero structure as E_k. By direct extension this holds equivalently for X_k

$$\begin{aligned}X_{k+1}=X_k+F_k(I-X_kY_k)^{-1}X_kE_k&=\left[\begin{array}{c|c}\underset{n^d\times n^-}{\overline{X_k}} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]+F_k(I-X_kY_k)^{-1}X_k\left[\begin{array}{c|c}\underset{n^d\times n^-}{\overline{E_k}} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]\\&=\left[\begin{array}{c|c}\underset{n^d\times n^-}{\overline{X_k}} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]+\left[\begin{array}{c|c}\underset{n^d\times n^-}{F_k(I-X_kY_k)^{-1}X_k\overline{E_k}} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]\\&=\left[\begin{array}{c|c}\underset{n^d\times n^-}{\overline{X_k}+F_k(I-X_kY_k)^{-1}X_k\overline{E_k}} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]=\left[\begin{array}{c|c}\underset{n^d\times n^-}{\overline{X_{k+1}}} & \underset{n^d\times n^{++}}{\mathbf{0}}\end{array}\right]\end{aligned}$$

Now for F_{k+1}

$$F_{k+1}=F_k(I-X_kY_k)^{-1}F_k=F_k(I-X_kY_k)^{-1}\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{\overline{F_k}}\end{array}\right]=\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{F_k(I-X_kY_k)^{-1}\overline{F_k}}\end{array}\right]=\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{\overline{F_{k+1}}}\end{array}\right]$$

By direct extension this holds equivalently for Y_k

$$\begin{aligned}Y_{k+1}=Y_k+E_k(I-Y_kX_k)^{-1}Y_kF_k&=\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{\overline{Y_k}}\end{array}\right]+E_k(I-Y_kX_k)^{-1}Y_k\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{\overline{F_k}}\end{array}\right]\\&=\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{\overline{Y_k}}\end{array}\right]+\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{E_k(I-Y_kX_k)^{-1}Y_k\overline{F_k}}\end{array}\right]\\&=\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{\overline{Y_k}+E_k(I-Y_kX_k)^{-1}Y_k\overline{F_k}}\end{array}\right]=\left[\begin{array}{c|c}\underset{n^d\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{\overline{Y_{k+1}}}\end{array}\right]\end{aligned}$$

This gives recursions in the generically non zero matrices $\overline{X_k}$, $\overline{Y_k}$, $\overline{E_k}$, and $\overline{F_k}$.

Noting that X_k and Y_k can be written out blockwise as

$$X_k=\begin{array}{cc} & \begin{array}{ccc} n^{--} & n^m & n^{++}\end{array}\\ \begin{array}{c} n^{--}\\ n^m\\ n^{++}\end{array} & \left[\begin{array}{c|c|c} X_{k;--,--} & X_{k;--,m} & \mathbf{0}\\ \hline X_{k;m,--} & X_{k;m,m} & \mathbf{0}\\ \hline X_{k;++,--} & X_{k;++,m} & \mathbf{0}\end{array}\right]\end{array}=\begin{array}{cc} & \begin{array}{ccc} n^{--} & n^m & n^{++}\end{array}\\ \begin{array}{c} n^{--}\\ n^m\\ n^{++}\end{array} & \left[\begin{array}{c|c} & \mathbf{0}\\ \underset{n^d\times n^-}{\overline{X_k}} & \mathbf{0}\\ & \mathbf{0}\end{array}\right]\end{array}=\begin{array}{cc} & \begin{array}{ccc} n^{--} & n^m & n^{++}\end{array}\\ \begin{array}{c} n^{--}\\ n^m\\ n^{++}\end{array} & \left[\begin{array}{c|c} \underset{n^{--}\times n^-}{\overline{\overline{X_k}}} & \mathbf{0}\\ \hline & \mathbf{0}\\ \underset{n^+\times n^-}{\overline{X_k}} & \mathbf{0}\end{array}\right]\end{array}$$

and

$$Y_k=\begin{array}{cc} & \begin{array}{ccc} n^{--} & n^m & n^{++}\end{array}\\ \begin{array}{c} n^{--}\\ n^m\\ n^{++}\end{array} & \left[\begin{array}{c|c|c} \mathbf{0} & Y_{k;--,m} & Y_{k;--,++}\\ \hline \mathbf{0} & Y_{k;m,m} & Y_{k;m,++}\\ \hline \mathbf{0} & Y_{k;++,m} & Y_{k;++,++}\end{array}\right]\end{array}=\begin{array}{cc} & \begin{array}{ccc} n^{--} & n^m & n^{++}\end{array}\\ \begin{array}{c} n^{--}\\ n^m\\ n^{++}\end{array} & \left[\begin{array}{c|c} \mathbf{0} & \\ \mathbf{0} & \underset{n^d\times n^+}{\overline{Y_k}}\\ \mathbf{0} & \end{array}\right]\end{array}=\begin{array}{cc} & \begin{array}{ccc} n^{--} & n^m & n^{++}\end{array}\\ \begin{array}{c} n^{--}\\ n^m\\ n^{++}\end{array} & \left[\begin{array}{c|c} \mathbf{0} & \\ \mathbf{0} & \underset{n^-\times n^+}{\overline{\overline{Y_k}}}\\ \hline \mathbf{0} & \underset{n^{++}\times n^+}{\overline{Y_k}}\end{array}\right]\end{array}$$

and hence their products can be calculated as

$$X_kY_k = \begin{array}{c} \\ n^{--} \\ n^{m} \\ n^{++} \end{array}\!\!\begin{array}{ccc} n^{--} & n^{m} & n^{++} \\ \left[\begin{array}{c|c} \mathbf{0} & \\ \mathbf{0} & \underset{n^d\times n^+}{\overline{X_k}\,\overline{\overline{Y_k}}} \\ \mathbf{0} & \end{array}\right] \end{array} \quad \text{and } Y_kX_k = \begin{array}{c} \\ n^{--} \\ n^{m} \\ n^{++} \end{array}\!\!\begin{array}{ccc} n^{--} & n^{m} & n^{++} \\ \left[\begin{array}{c|c} & \mathbf{0} \\ \underset{n^d\times n^-}{\overline{Y_k}\,\underline{\overline{X_k}}} & \mathbf{0} \\ & \mathbf{0} \end{array}\right] \end{array}$$

Similar calculations apply to $Y_k\overline{F_k}$ and $X_k\overline{E_k}$.

For Algorithm 1 with a nonzero P_0, note that if $\widehat{X}_0$, $\widehat{Y}_0$, $\widehat{E}_0$, $\widehat{F}_0$ have (zero and non zero) dimensions that correspond to those of X_0, Y_0, E_0, and F_0, the same approach can be taken. Comparing, the only requirement is that P_0 takes the form

$$P_0 = \begin{array}{c} \\ n^{--} \\ n^{m} \\ n^{++} \end{array}\!\!\begin{array}{ccc} n^{--} & n^{m} & n^{++} \\ \left[\begin{array}{c|c} & \mathbf{0} \\ \underset{n^{--}\times n^-}{\overline{P_0}} & \mathbf{0} \\ & \mathbf{0} \end{array}\right] \end{array}$$

which conforms to the structure of the matrix quadratic as shown in the subsection above.

7. Detailed Dynare Topology - SDA for SF2

Beginning with Algorithm 2 with an initial guess $P_0 = 0$, the initial values

$$X_0^\dagger = 0, \qquad Y_0^\dagger = -B, \qquad E_0^\dagger = -C, \qquad F_0^\dagger = -A$$

can be written as

$$X_0^\dagger = 0, \qquad Y_0^\dagger = -\underset{n^d\times n^d}{\tilde{\mathbf{A}}^0}, \qquad E_0^\dagger = n^d \begin{array}{cc} n^- & n^{++} \\ \left[\begin{array}{c|c} \underset{n^d\times n^-}{-\tilde{\mathbf{A}}^-} & \mathbf{0} \end{array}\right] \end{array}, \qquad F_0^\dagger = n^d \begin{array}{cc} n^{--} & n^{+} \\ \left[\begin{array}{c|c} \mathbf{0} & \underset{n^d\times n^+}{-\tilde{\mathbf{A}}^+} \end{array}\right] \end{array}$$

For the calculations, we will work with:

$$E_{k+1}^\dagger = E_k^\dagger\left(X_k^\dagger - Y_k^\dagger\right)^{-1}E_k^\dagger$$
$$F_{k+1}^\dagger = F_k^\dagger\left(X_k^\dagger - Y_k^\dagger\right)^{-1}F_k^\dagger$$
$$X_{k+1}^\dagger = X_k^\dagger - F_k^\dagger\left(X_k^\dagger - Y_k^\dagger\right)^{-1}E_k^\dagger$$
$$X_{k+1}^\dagger - Y_{k+1}^\dagger = X_k^\dagger - Y_k^\dagger - F_k^\dagger\left(X_k^\dagger - Y_k^\dagger\right)^{-1}E_k^\dagger - E_k^\dagger\left(X_k^\dagger - Y_k^\dagger\right)^{-1}F_k^\dagger$$

As above, $E_{k+1}^\dagger$ and $F_{k+1}^\dagger$ will maintain the (zero and non zero) dimensions that correspond to $E_0^\dagger$ and $F_0^\dagger$ via the post multiplication of $E_k^\dagger$ and $F_k^\dagger$ and induction. The same post multiplication gives $X_{k+1}^\dagger$ the (zero and non zero) dimensions that correspond to $E_0^\dagger$

$$\begin{aligned} X_{k+1}^\dagger = X_k^\dagger - F_k^\dagger\left(X_k^\dagger - Y_k^\dagger\right)^{-1}E_k^\dagger &= \left[\begin{array}{c|c} \underset{n^d\times n^-}{\overline{X_k}^\dagger} & \underset{n^d\times n^{++}}{\mathbf{0}} \end{array}\right] - F_k^\dagger\left(X_k^\dagger - Y_k^\dagger\right)^{-1}\left[\begin{array}{c|c} \underset{n^d\times n^-}{\overline{E_k}^\dagger} & \underset{n^d\times n^{++}}{\mathbf{0}} \end{array}\right] \\ &= \left[\begin{array}{c|c} \underset{n^d\times n^-}{\overline{X_k}^\dagger} & \underset{n^d\times n^{++}}{\mathbf{0}} \end{array}\right] - \left[\begin{array}{c|c} \underset{n^d\times n^-}{F_k^\dagger\left(X_k^\dagger - Y_k^\dagger\right)^{-1}\overline{E_k}^\dagger} & \underset{n^d\times n^{++}}{\mathbf{0}} \end{array}\right] \\ &= \left[\begin{array}{c|c} \underset{n^d\times n^-}{\overline{X_k}^\dagger - F_k^\dagger\left(X_k^\dagger - Y_k^\dagger\right)^{-1}\overline{E_k}^\dagger} & \underset{n^d\times n^{++}}{\mathbf{0}} \end{array}\right] = \left[\begin{array}{c|c} \underset{n^d\times n^-}{\overline{X_{k+1}}^\dagger} & \underset{n^d\times n^{++}}{\mathbf{0}} \end{array}\right] \end{aligned}$$

For Algorithm 2 with an initial guess P_0, $E_0^{\dagger}$ and $F_0^{\dagger}$ are unchanged, so the only requirement is that P_0 takes the form

$$P_0 = \begin{array}{c} \\ n^{--} \\ n^{m} \\ n^{++} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \quad n^{--} \quad n^{m} & \quad n^{++} \end{array} \\ \left[\begin{array}{c|c} & \mathbf{0} \\ \underset{n^{--}\times n^{-}}{\overline{P_0}} & \mathbf{0} \\ & \mathbf{0} \end{array}\right] \end{array}$$

to have (zero and non zero) dimensions that correspond to $E_0^{\dagger}$ - this conforms to the structure of the matrix quadratic as shown in the subsection above.

8. Smets and Wouters (2007) Parameterizations

Table 1 reports the two different paramterizations for Smets and Wouters (2007) in the main text. Alongside these parameters that are estimated in the original study, a number of parameters are also calibrated by the authors and are kept fixed here throughout.

TABLE 1. Smets and Wouters (2007) Parameterizations

Measure	Posterior Mode	Problematic
ρ_{ga}	0.5187	0.499734578
$\bar{l}$	0.5509	0.433312408
$\bar{\pi}$	0.7869	0.795582157
$100(\beta^{-1}-1)$	0.1657	0.141824381
μ_w	0.8503	0.839433664
μ_p	0.701	0.763719864
α	0.1901	0.997405167
ψ	0.5462	0.582979079
φ	5.7606	5.841219834
σ_c	1.3808	1.643185575
λ	0.7133	0.360646436
ϕ_p	1.6064	33.19042186
ι_w	0.5845	0.511066325
ξ_w	0.7061	0.381966011
ι_p	0.2432	0.854101966
ξ_p	0.6523	0.797807656
σ_l	1.8383	1.120655188
r_π	2.0443	53.35560742
$r_{\Delta y}$	0.2247	-0.318651967
r_y	0.0882	1.673342044
ρ	0.8103	0.616500961
ρ_a	0.9577	0.204061725
ρ_b	0.2194	0.159354861
ρ_g	0.9767	0.650117912
ρ_i	0.7113	0.730155241
ρ_r	0.1479	0.4485053
ρ_p	0.8895	0.909998197
ρ_w	0.9688	0.604705301
$\bar{\gamma}$	0.4312	0.435912132

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